2.3: Conditional Statements

Consider the statement

R: If the shape is a circle of radius 3, then it has area 9π.

The statement R is built out of the smaller statements:

P: The shape is a circle of radius 3
Q: The shape has area 9π

R: If P, then Q

We write R symbolically as $P \Rightarrow Q$
This type of statement is called a conditional statement because it means that Q will be true under the condition that P is true.

What does a truth table look like?

So far, if P & Q are true, we know $P \Rightarrow Q$ is true.

30

If P is true and Q is false, then a circle of radius 3 would have area other than 9π. This is a <u>false</u> statement.

The next two are trickier: ① P false, Q true
This would be "The shape is not a circle of radius 3, but the shape has area 9π."
This is <u>true</u> because a shape does not need to be a circle of radius 3 to have area 9π.

② P false, Q false
"The shape is not a circle of radius 3 and does not have area 9π."
$P \Rightarrow Q$ is still true, because if a shape isn't a circle of radius 3, then there's no reason for it to have area 9π.

This gives the truth table:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

2.4: Biconditional Statements

A biconditional statement is an "if and only if" statement. It is a combination of $P \Rightarrow Q$ and its converse $Q \Rightarrow P$. In general, the converse is not true, for example, if

P: You don't eat your meat

Q: You can't have any pudding

We all know that $P \Rightarrow Q$, but $Q \Rightarrow P$ is not necessarily true... you may eat your meat, but be lactose intolerant and so cannot eat pudding!

So this is not a biconditional statement.

On the other hand, if

P: The circle has radius r

Q: The circle has area πr^2

$P \Rightarrow Q$: If the circle has radius r, then the circle has area πr^2 True

$Q \Rightarrow P$: If the circle has area πr^2, then the circle has radius r. True

So, we get the biconditional statement

$P \Leftrightarrow Q$: The circle has radius r if and only if it has area πr^2.

Since $P \Rightarrow Q$ and $Q \Rightarrow P$ in a biconditional statement, we write $P \Leftrightarrow Q$.

The truth table can be seen as
$$P \Leftrightarrow Q = (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q = (P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

2.5: Truth Tables for Statements

We have already touched on the topics of, but let's do some more examples:

1. Come up with a statement for the exclusive or: "P or Q, but not both"

"P or Q" $\rightarrow P \vee Q$

"but" $\longrightarrow \wedge$

"not both" $\longrightarrow \sim(P \wedge Q)$

We will want True only when P & Q have opposite values.

33

So, perhaps $(P \vee Q) \wedge \sim(P \wedge Q)$ will work:

P	Q	$P \vee Q$	$P \wedge Q$	$\sim(P \wedge Q)$	$(P \vee Q) \wedge \sim(P \wedge Q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

So $(P \vee Q) \wedge \sim(P \wedge Q)$ is the "exclusive" or

Ex: Construct a truth table for

$$[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$$

P	Q	R	$P \Rightarrow Q$	$Q \Rightarrow R$	$(P \Rightarrow Q) \wedge (Q \Rightarrow R)$	$P \Rightarrow R$	$[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

This is an example of a tautology: a statement which is true in every interpretation.